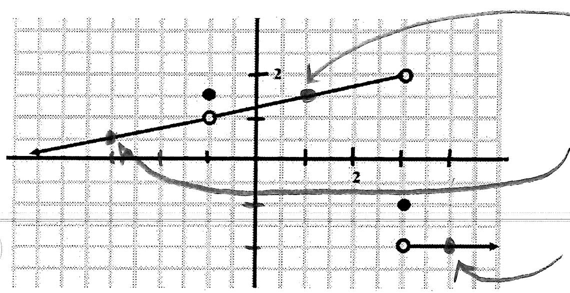
Chapter 2 Notes　　　　　　　　　　Name Key (2014)

2.1 (part I): Limits - using graphs

Limit: What y value is approached as x approaches a certain value

- ***notation:*** $\lim_{x \to c} f(x) = L$ "The limit of f(x) as x approaches c is equal to L"

Example:



$\lim_{x\to 1} f(x) = 1.5$

$\lim_{x\to -3} f(x) = 0.5$

$\lim_{x\to 4} f(x) = -2$

One-Sided Limits:

- Right-hand limit: $\lim_{x\to c^+} f(x)$ → limit as x approaches c from the right
- Left-hand limit: $\lim_{x\to c^-} f(x)$ → limit as x approaches c from the left

Theorem: The limit of f(x) as x approaches c *exists* iff the right-hand limit and the left-hand limit at c exist and are equal.

$\lim_{x\to c} f(x) = L$ iff $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

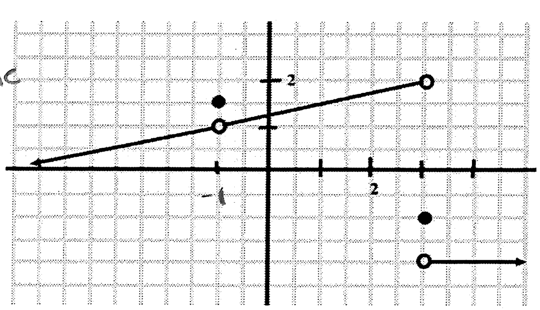
$\lim_{x\to -1^-} f(x) = 1$

$\lim_{x\to -1^+} f(x) = 1$

same

$\lim_{x\to -1} f(x) = 1$

however: $f(-1) = 1.5$



$\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^-} f(x) = -2$

different

$\lim_{x\to 3} f(x) =$ DNE

$f(3) = -1$

1. $\lim_{x\to-3} f(x) = 2$

$f(-3) = 1$

$\lim_{x\to-1} f(x) = 3$

$f(-1) = \text{DNE}$

$f(1) = 2$

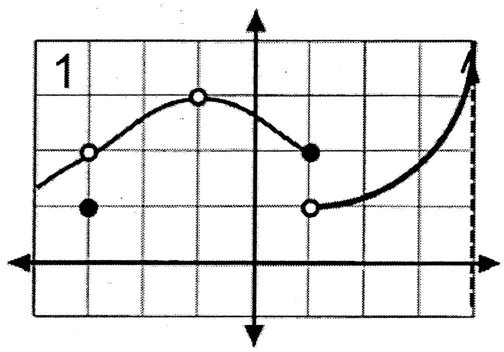
$\lim_{x\to1^-} f(x) = 2$

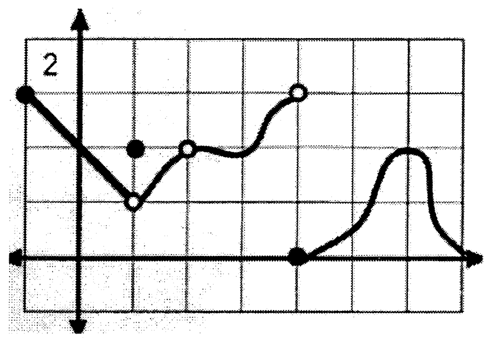
$\lim_{x\to1^+} f(x) = 1$

$\lim_{x\to1} f(x) = \text{DNE}$

$\lim\limits_{x\to4} f(x) = \infty$

(DNE)





2. $\lim_{x\to2} f(x) = 2$

$f(1) = 2$

$f(-1) = 3$

$\lim_{x\to1} f(x) = 1$

$\lim_{x\to0} f(x) = 2$

$\lim_{x\to4^-} f(x) = 3$

$\lim_{x\to4^+} f(x) = 0$

$\lim_{x\to4} f(x) = \text{DNE}$

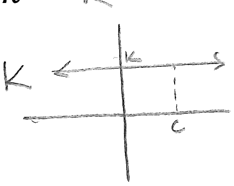
2.1 (part II): Limits - algebraically

Properties of Limits

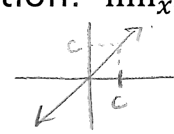
Given: $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

1. Sum/Difference: $\lim_{x\to c}(f(x) \pm g(x)) = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
2. Product: $\lim_{x\to c}(f(x) \cdot g(x)) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$
3. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$
4. Constant Multiple: $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim\limits_{x\to c} f(x)$
5. Function with a Constant Value: $\lim_{x\to c} k = k$
6. Identity Function: $\lim_{x\to c} x = c$

$y = k$



$y = x$:



Finding Limits Analytically: try these methods in this order

*canceling factor in num. & denom. removes hole in graph

I. Substitute:

$\lim_{x\to -2}(x-6)^{2/3}$

$= (-2-6)^{2/3} = (-8)^{2/3}$

$= \left((-8)^{1/3}\right)^2 = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = \boxed{4}$

$\lim_{x\to 0}\left(x^3 - \frac{\cos 5x}{10}\right)$

$= 0^3 - \frac{\cos(0)}{10}$

$= 0 - \frac{1}{10} = \boxed{-1/10}$

$\lim_{x\to 0}\left(\frac{\sin(2-x)}{2-x}\right)$

$= \boxed{\frac{\sin(2)}{2}}$

II. Factor:

$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$

function has hole @ $x=2$

function same as orig. w/out the hole

$= \lim_{x\to 2}(x+2) = 2+2 = \boxed{4}$

$\lim_{x\to 3}\left(\frac{x^3-27}{x-3}\right) = \lim_{x\to 3}\frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}}$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & 0 & -27 \\ & & 3 & 9 & 27 \\ \hline & 1 & 3 & 9 & \vert\ 0 \end{array}$$

$\underbrace{1x^2+3x+9}$ remainder

$= \lim_{x\to 3}(x^2+3x+9)$

$= 3^2 + 3\cdot 3 + 9$

$= \boxed{27}$

III. "Simplify": change to another form

$$\lim_{x\to 0}\frac{(x+3)^2-9}{x} = \lim_{x\to 0}\frac{x^2+6x+9-9}{x} = \lim_{x\to 0}\frac{x^2+6x}{x} = \lim_{x\to 0}\frac{\cancel{x}(x+6)}{\cancel{x}}$$

$$= \lim_{x\to 0}(x+6) = 0+6 = \boxed{6}$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x\to 0}\frac{\cancel{x^2}+\cancel{9}+\cancel{3\sqrt{x^2+9}}-\cancel{3\sqrt{x^2+9}}-\cancel{9}}{x^2\left(\sqrt{x^2+9}+3\right)}$$

multiply by "conjugate"

$$= \lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+9}+3\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

IV. Special Cases:

$$\lim_{x\to 0}\frac{|x|}{x}$$

left: $\lim_{x\to 0^-}\frac{|x|}{x} \approx \frac{|-.0001|}{-.0001} = \frac{.0001}{-.0001} = -1$

right: $\lim_{x\to 0^+}\frac{|x|}{x} \approx \frac{|.0001|}{.0001} = \frac{.0001}{.0001} = 1$

different: shows f(x) has jump @ x=0

$\lim_{x\to 0}\frac{|x|}{x} = \boxed{DNE}$

$$\lim_{x\to 0}\frac{\sin x}{x} \approx \frac{\sin(.0001)}{.0001}$$?? not sure. Use table on calculator.

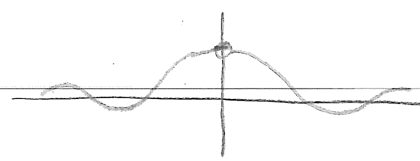
graph: $y_1 = \frac{\sin x}{x}$

tableset: TblStart: $x=0$

ΔTbl $= .0001$ ← tiny #

x	y
-.0002	1
-.0001	1
0	error
.0001	1
.0002	1

values so close to 1, calc. rounds to 1



Examples:

$\lim_{a\to -9}\frac{a^2+4a+3}{a^2-9}$

$= \frac{(-9)^2+4(-9)+3}{(-9)^2-9} = \frac{48}{72} = \boxed{\frac{2}{3}}$

$\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1}\frac{1}{x+1}$

has "hole" at x=1

doesn't have hole any more

$= \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$\lim_{z\to -2}\frac{2+z}{\sin(2+z)} = 1$

$\lim_{x\to \square}\frac{\sin(\ \ )}{(\ \ )} = 1$

must be same and equal zero when x plugged in

$\lim_{x\to 0}\frac{x+\sin x}{x} = \lim_{x\to 0}\left(\frac{x}{x}+\frac{\sin x}{x}\right)$

$= \lim_{x\to 0}(1) + \lim_{x\to 0}\left(\frac{\sin x}{x}\right)$

$= 1+1$

$= \boxed{2}$

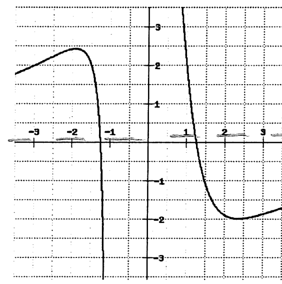
2.2: Asymptotes

(In Alg. & Trig. we did this.)

For a rational function, Horizontal Asymptotes are found using the degree of the numerator (n) and the degree of the denominator (m):

1. If n < m , then y = 0 is a horizontal asymptote

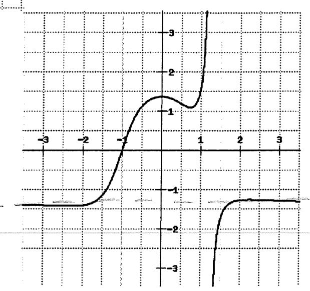
$$y = \frac{-7x^{2} + 11}{x^{3} + 1}$$



2. If n = m, then y = ratio of the leading coefficients

$$y = \frac{4x^{5} - 7x^2 + 11}{8 - 3x^{5}}$$

$y = \frac{4}{-3}$ HA



$y = -4/3$

Note: If n > m, then there is a slant asymptote (no H.A.)

$$y = \frac{x^{2} - 2x - 15}{x^{1} + 4}$$

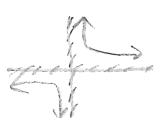
no HA

Definition: Horizontal Asymptote

If either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, then

$y = b$ is horizontal asymptote (HA)

$y = \frac{1}{x} \rightarrow$



Example: Using definition to find H.A.

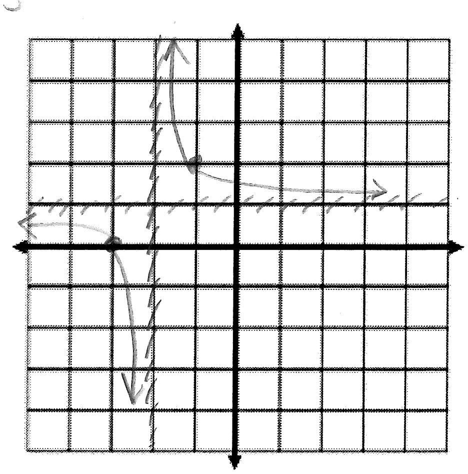
alg. method

$$f(x) = \frac{1}{x+2} + 1 = \frac{1}{x+2} + \frac{x+2}{x+2} = \frac{x+3}{x+2}$$

up 1

left 2

$\boxed{y = 1}$ HA



$$\lim_{x\to+\infty} f(x) = \frac{1}{\infty + 2} + 1 = \frac{1}{\text{huge}} + 1$$

$$= 0 + 1 = 1$$

$$\lim_{x\to-\infty} f(x) = \frac{1}{-\infty + 2} + 1 = \frac{1}{-\text{huge}} + 1$$

$$= 0 + 1 = 1$$

by definition: HA $y = 1$

- **End Behavior Model** use a function ("simpler") that behaves same way at "ends" as given (more complex) function. EBM = $\frac{\text{term with highest degree}}{\text{term with highest degree}}$

Example: Use definition to find H.A.

$$f(x) = \frac{x^5 - 7x^2 + 11}{8 - 3x^5}$$

$$\lim_{x\to\infty} \frac{x^5 - 7x^2 + 11}{8 - 3x^5} \underset{\text{EBM}}{=} \lim_{x\to\infty} \frac{x^5}{-3x^5} = \frac{-1}{3} \Rightarrow \boxed{\text{HA: } y = \frac{-1}{3}}$$

$$\lim_{x\to-\infty} \frac{x^5}{-3x^5} = \frac{-1}{3}$$

$$f(x) = \left[\left(2 - \frac{x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right)\right]$$

$$\lim_{x\to\infty} \left[\left(2 - \frac{x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right)\right] \underset{\text{EBM}}{=} \lim_{x\to\infty} \left[\left(2 - \frac{x}{x}\right)\left(\frac{x^2}{x^2}\right)\right]$$

$$= \lim_{x\to\infty} (2-1)(1) = 1 \qquad \boxed{\text{HA: } y = 1}$$

$$f(x) = \frac{\sin x}{x}$$

$$\lim_{x\to\infty} \frac{\sin x}{x} = \frac{\sin \infty}{\infty}$$ EBM?? Nope $$\lim_{x\to-\infty} \frac{\sin x}{x} = \frac{\sin(-\infty)}{-\infty}$$

$$= \frac{[-1,1]}{\infty} = 0 \qquad = \frac{[-1,1]}{-\infty} = 0$$

$$\boxed{\text{HA: } y = 0}$$

$$f(x) = \frac{3x+1}{|x|+2}$$

$$\lim_{x\to\infty} \frac{3x+1}{|x|+2} \underset{\text{EBM}}{=} \lim_{x\to\infty} \frac{3x}{|x|} = \lim_{x\to\infty} \frac{3(\infty)}{|\infty|} = 3(1) = 3$$

$$\lim_{x\to-\infty} \frac{3x}{|x|} = \frac{3(-\infty)}{|-\infty|} = 3(-1) = -3$$

$$\boxed{\begin{array}{l}\text{HA: } y = 3 \\ \phantom{\text{HA: }} y = -3\end{array}}$$

Example: Use graphing calculator to find H.A. for $f(x) = \frac{3x+1}{|x|+2}$

$Y_1 = (3x+1)/(\text{abs}(x)+2)$

Tblset: Start: $x = \pm 9999$ ← big #

$\Delta x = 100$ ← big-ish #

For a rational function, Vertical Asymptotes are found where the function is undefined. (Caution: may be a **hole** in the graph) Factor first!

Alg. method

$$y = \frac{x^2 - 2x - 15}{x + 4} = \frac{(x-5)(x+3)}{x+4}$$

If $x = -4$, $f(x) = $ DNE

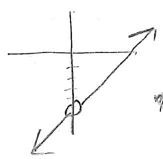
$\boxed{x = -4}$ VA

$$y = \frac{x^2 - 2x - 15}{x + 3} = \frac{(x-5)\cancel{(x+3)}}{\cancel{x+3}} = x - 5$$

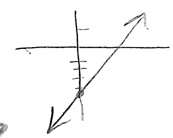
no V.A.

$\boxed{\text{hole @ } x = -3}$

$(-3, -8)$



same graph but this has hole, this doesn't



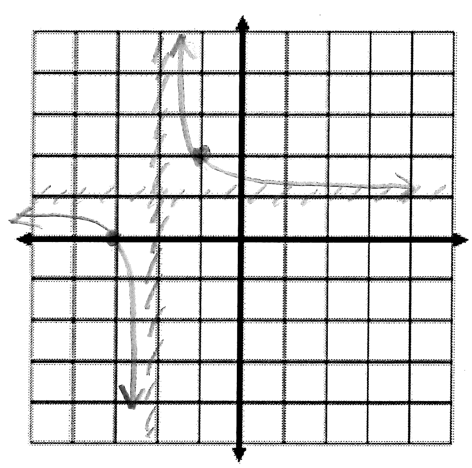
Definition: Vertical Asymptote

If either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$, then

$x = a$ is vertical asymptote (VA)

Example: Using definition to verify V.A.

$$f(x) = \frac{1}{x+2} + 1$$



$$\lim_{x \to -2^+} f(x) = \frac{1}{-1.999 + 2} + 1 = \frac{1}{.001} + 1 = \infty$$

$$\lim_{x \to -2^-} f(x) = \frac{1}{-2.001 + 2} + 1 = \frac{1}{-.001} + 1 = \infty$$

V.A. $\boxed{x = -2}$

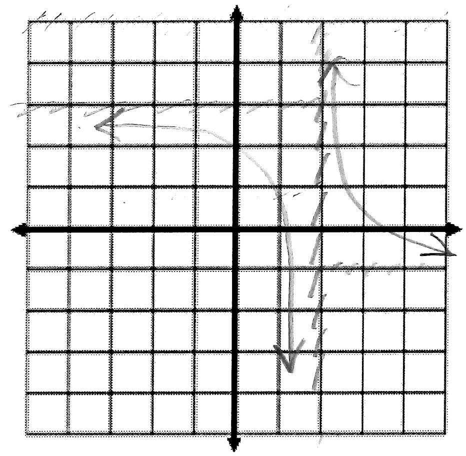
Example: Sketch a function that meets the criteria.

$\lim_{x \to +\infty} f(x) = -1$ HA: y=−1 (right end)

$\lim_{x \to -\infty} f(x) = 3$ HA: y= 3 (left end)

$\lim_{x \to 2^+} f(x) = \infty$ VA: x= 2

$\lim_{x \to 2^-} f(x) = -\infty$ VA: x= 2



Example: Sketch a function that meets the criteria.

$g(-3) = 4$ → point at (−3, 4)

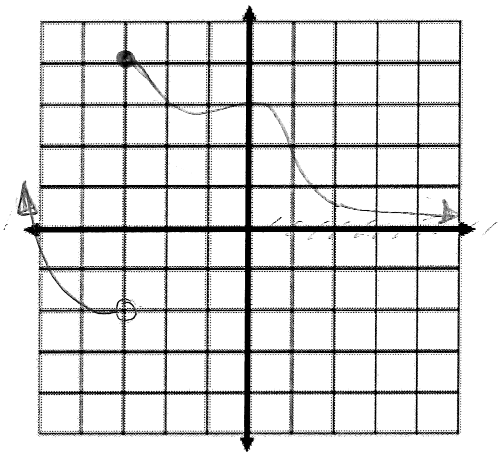
$\lim_{x \to -3^-} g(x) = -2$ y approaches −2 left of x=−3

$\lim_{x \to -3^+} g(x) = 4$ y approaches 4 right of x=−3

jump

$\lim_{x \to -\infty} g(x) = \infty$ no HA, on left end

$\lim_{x \to +\infty} g(x) = 0$ HA y=0 on right end



2.3: Continuity

Where is the function **discontinuous**?

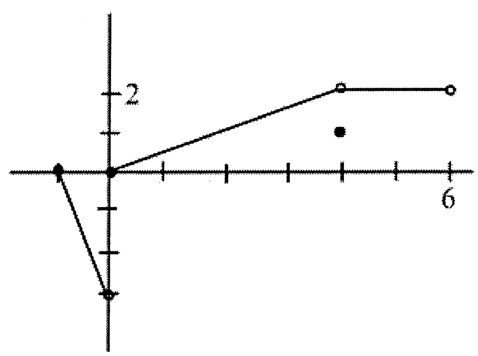
not equal

jump: x=0 → $\lim_{x \to 0} f(x) =$ DNE , f(0)=0

hole: x=4 , x=6 → $\lim_{x \to 4} f(x) = 2$, f(4)=1

VA: none

not equal



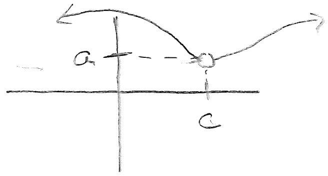
Where is the function **continuous**?

continuous for all x, x ≠ 0, 4, 6

Definition: $f(x)$ is continuous at an interior point c if $\lim_{x \to c} f(x) = f(c)$

Four Types of discontinuity:

- point (hole)

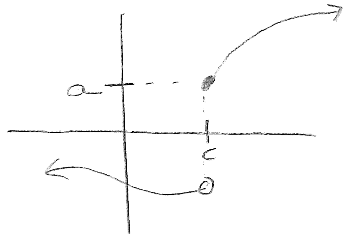


$\lim_{x \to c} f(x) = a$

$f(c) = DNE$ ← f(x) has no y value at x=c

not same ⇒ discontinuous

- jump



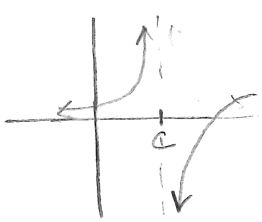
left lim ≠ right lim

$\lim_{x \to c} f(x) = DNE$

$f(c) = a$

not same ⇒ discontinuous

- infinite (vertical asymptote)

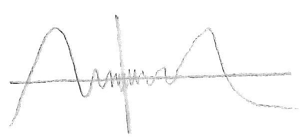


$\lim_{x \to c^-} f(x) = +\infty$ $\lim_{x \to c^+} f(x) = -\infty$

$f(c) = DNE$

- oscillating crazy!!

ex) $y = \sin\left(\frac{1}{x}\right)$ DNE @ x = 0, but not a hole or VA or jump



- closer you get to x = 0, more the function oscillates (zoom in on calc)

Example: Find each discontinuity and classify ~~each using limits.~~ which type.

$$f(x) = \frac{2x+5}{x-4}$$

$\lim_{x \to 4^+} f(x) = \frac{2(4.001)+5}{4.001-4} = \frac{8.002+5}{.001} = \frac{13.002}{.001} \to \infty$ smaller, smaller

f(x) DNE @ x = 4

$\lim_{x \to 4^-} f(x) = \frac{2(3.999)+5}{3.999-4} = \frac{13\text{ish}}{-.001} \to -\infty$

$\boxed{VA @ x = 4}$

$$f(x) = \frac{x^3 - 7x - 6}{x^2 - 9} = \frac{(x-3)(x^2+3x+2)}{(x-3)(x+3)}$$

$$= \frac{(x+1)(x+2)}{x+3}$$

$\boxed{x = 3 \text{ pt. disc.}}$

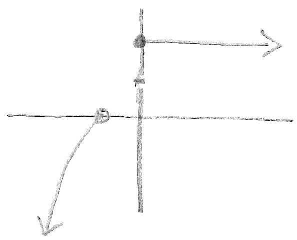
$\boxed{x = -3 \text{ V.A.}}$

3	1	0	−7	−6
		3	9	6
	1	3	2	0

$$f(x) = \begin{cases} 1 - x^2, & x < -1 \\ 2, & x \geq -1 \end{cases}$$

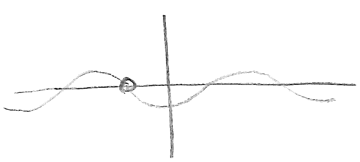
$\lim_{x \to -1^-} f(x) = 1-(-1)^2 = 0$

$\lim_{x \to -1^+} f(x) = 2 \leftarrow$ different, so $\boxed{\text{jump @} x = -1}$



$$f(x) = \cos\left(\frac{x^2 - 9}{x+3}\right) = \cos\left(\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}\right)$$

$\boxed{x = -3 \text{ pt. disc}}$



$f(-3) = \text{DNE}$

$\lim_{x \to -3} f(x) = \cos(-6)$

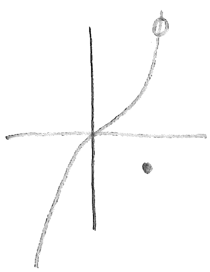
$$f(x) = \begin{cases} x^3, & x \neq 2 \\ -3, & x = 2 \end{cases}$$

jump? hole?

$\lim_{x \to 2} f(x) = 8$

$f(2) = -3$

$\boxed{x = 2 \text{ pt. disc}}$



Extended functions: The "after" function once its hole is removed

Example: Give a formula for the extended function. Then assign a y value for the point of discontinuity for the original function.

same, but w/out the hole

$$f(x) = \frac{x^2 + 2x - 15}{2x^2 + 10x} = \frac{\cancel{(x+5)}(x-3)}{2x\cancel{(x+5)}} \Rightarrow \boxed{y = \frac{x-3}{2x}}$$

hole @ $x = -5$

"extended function"

$y(-5) = \frac{-5-3}{-5 \cdot 2} = \frac{-8}{-10}$

$\boxed{y = 4/5}$

this y value would fill in hole @ $x = -5$

$$f(x) = \frac{x^2 - 36}{x + 6} = \frac{\cancel{(x+6)}(x-6)}{\cancel{x+6}}$$

extended function

$\boxed{y = x - 6}$

hole @ $x = -6$

$y(-6) = -6 - 6$

$\boxed{y = -12}$

this y value will fill in hole @ $x = -6$

$$f(x) = \frac{x^4 - 3x^2 - 4}{x - 2} = \frac{(x^2-4)(x^2+1)}{x-2}$$

$$= \frac{(x+2)\cancel{(x-2)}(x^2+1)}{\cancel{x-2}}$$

extended function

$\boxed{y = (x+2)(x^2+1)}$

$y(2) = (2+2)(2^2+1)$

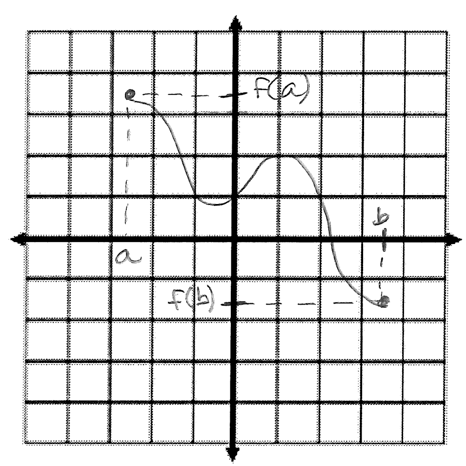
$y = 4 \cdot 5$

$\boxed{y = 20}$ this y value fills in hole @ $x = 2$

hole @ $x = 2$

Intermediate Value Theorem (IVT)

If the function y = f(x) is continuous on [a,b], then f(x) takes on all y values between f(a) and f(b).



speed = $\dfrac{\text{change of distance}}{\text{change of time}}$

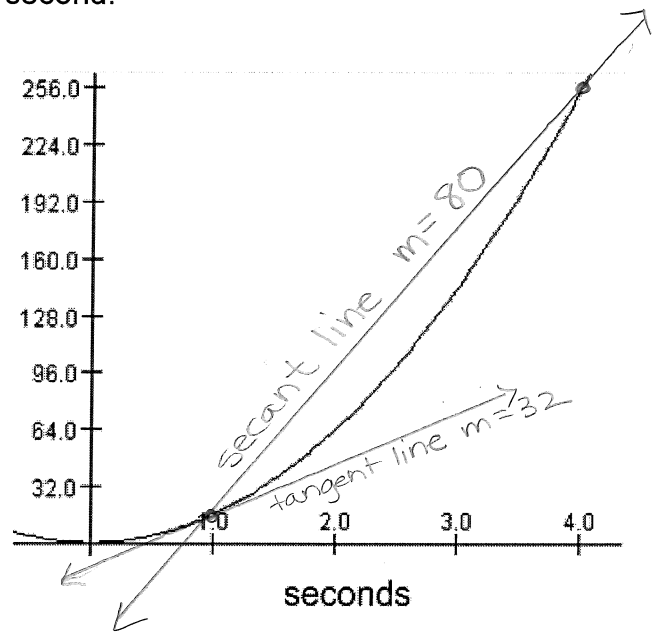
2.4: Rate of Change and Tangent Lines

The equation for free fall of a dense object is $y = 16t^2$ feet (t in seconds).

- Find the average speed between the 1st and 4th second.

Ave. speed $= \dfrac{y(4) - y(1)}{4 - 1} \dfrac{\text{ft}}{\text{sec}}$

$= \dfrac{256 - 16}{3} = \dfrac{240}{3} = \boxed{80 \text{ ft/sec}}$



- Find the instantaneous speed at t = 1

estimate $\dfrac{y(1.1) - y(1)}{1.1 - 1} = \dfrac{19.36 - 16}{.1} = \dfrac{3.36}{.1}$

$= 33.6$ ft/sec

better estimate $\dfrac{y(1.01) - y(1)}{1.01 - 1} = \dfrac{16.3216 - 16}{.01}$

$= 32.16$ ft/sec

Need a limit!

Instantaneous speed $= \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{1+h-1} = \lim_{h \to 0} \dfrac{16(1+h)^2 - 16(1)^2}{h} = \lim_{h \to 0} \dfrac{16(1+2h+h^2) - 16}{h}$

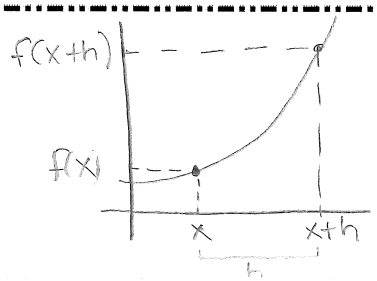
$= \lim_{h \to 0} \dfrac{16 + 32h + 16h^2 - 16}{h} = \lim_{h \to 0} \dfrac{32h + 16h^2}{h} = \lim_{h \to 0} 32 + 16h = \boxed{32 \text{ ft/sec}}$

slope of secant

Ave. rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(b) - f(a)}{b - a}$ "Algebra Slope"

slope of tangent

Instantaneous rate of change $= \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ "Calculus slope"



ave. rate of change $= \dfrac{f(x+h) - f(x)}{x+h-x}$

instantaneous rate of change $= \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{x+h-x}$

Example: $f(x) = x^2 - 4x$

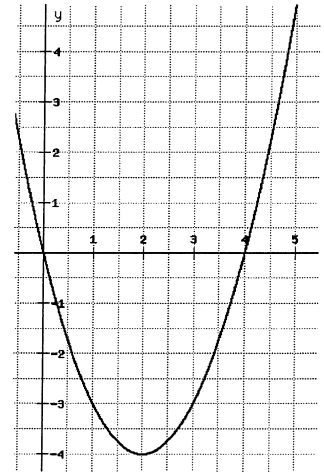
- Find the instantaneous rate of change at $x = a$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2-4(a+h)-[a^2-4a]}{h}$$

$$= \lim_{h\to 0} \frac{a^2+2ah+h^2-4a-4h-a^2+4a}{h} = \lim_{h\to 0} \frac{2ah+h^2-4h}{h}$$

$$= \lim_{h\to 0} \frac{h(2a+h-4)}{h} = \lim_{h\to 0} (2a+h-4) = \boxed{2a-4}$$

$x=a$



- What is the slope at $x = 1$?

$m = 2(1) - 4$

$\boxed{m = -2}$

- Where is the slope of a tangent to the graph of f(x) equal to 6?

$2a - 4 = 6$

$2a = 10$

$a = 5$

$\boxed{x = 5}$

- What is the **equation of the tangent** line at the point when $m = 6$?

$(5,5)$

$f(5) = 5^2 - 4\cdot 5$

$= 25 - 20 = 5$

$\boxed{y - 5 = 6(x-5)}$

- What is the **equation of the normal** line at the point when $m = 6$?

can only do this (just change slope) when in point-slope form

$\boxed{y - 5 = -\frac{1}{6}(x-5)}$

Example: $f(x) = \begin{cases} x^3 + 2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ this part

check for jump first: (must be continuous)

left $f(0) = 0^3 + 2 = 2$

right $f(0) = 2\cdot 0 + 2 = 2$

no jump → now must check both slopes. They must be same at "connection" or slope DNE there.

Determine the slope of the curve at $x = 3$ and at $x = 0$ (if they exist).

$$\lim_{h\to 0} \frac{2(3+h)+2-(2\cdot 3+2)}{h} = \lim_{h\to 0} \frac{6+2h+2-6-2}{h} = \lim_{h\to 0} \frac{2h}{h} = \boxed{2}$$

left: $\lim_{h\to 0} \frac{(h+0)^3+2-(0^3+2)}{h} = \lim_{h\to 0} \frac{h^3+2-2}{h} = \underline{\underline{0}}$

not same

right: $\lim_{h\to 0} \frac{2(0+h)+2-(2\cdot 0+2)}{h} = \lim_{h\to 0} \frac{2h+2-2}{h} = \lim_{h\to 0} \frac{2h}{h} = \underline{\underline{2}}$

so derivative @ $x = 0$ for $f(x)$ DNE

instantaneous

Example: What is the rate of change (in/sec) for the area of an expanding circle when r = 3”?

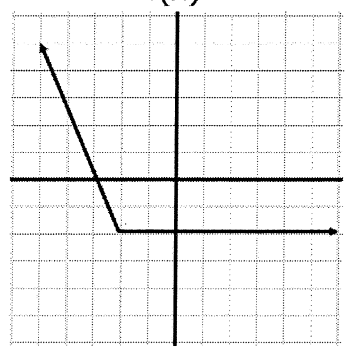
$A = \pi r^2$

$$\lim_{h\to 0} \frac{\pi(3+h)^2 - \pi\cdot 3^2}{h} = \lim_{h\to 0} \frac{\pi(9+6h+h^2) - \pi\cdot 9}{h}$$

$$= \lim_{h\to 0} \frac{\pi\cdot 9 + 6h\pi + h^2\pi - \pi\cdot 9}{h} = \lim_{h\to 0}$$

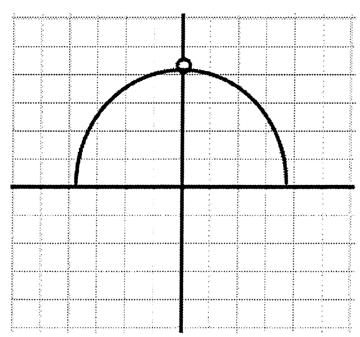
Summary: Limits…Continuity…Slope of curve

f(x)



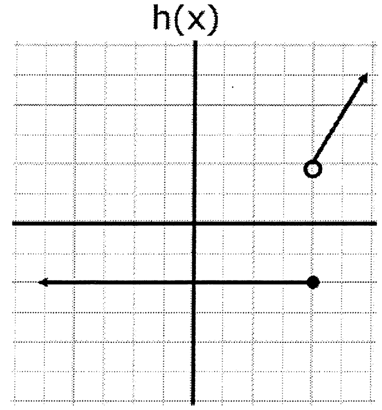
- $\lim_{x\to -1} f(x) =$ −1
- Is $f(x)$ continuous at $x = -1$? yes
  no holes, jumps, VA
- Slope of $f(x)$ at $x = -1$? DNE
  2 diff. slopes there

g(x)



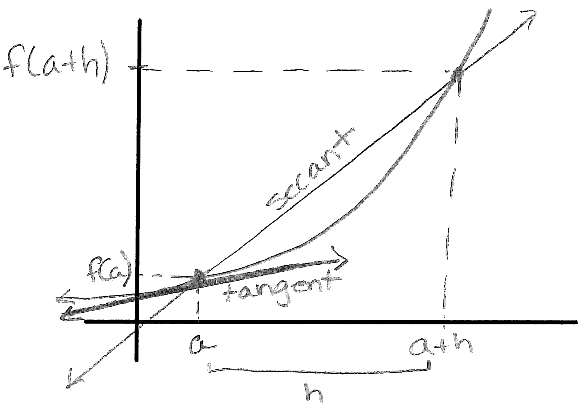
- $\lim_{x\to 0} g(x) =$ 2
- Is $g(x)$ continuous at $x = 0$? No, hole
- Slope of $g(x)$ at $x = 0$? DNE, because discontinuity @ x=0 (hole)

h(x)



- $\lim_{x\to 2} h(x) =$ DNE b/c 2 diff. y values
- Is $h(x)$ continuous at $x = 2$?
  no, jump
- Slope of $h(x)$ at $x = 2$? DNE
  b/c discontinuous @ x=2

3.1 Derivative of a Function



Definition: Derivative of f at a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, then f is "differentiable" at a

Notation for derivatives:

$f'(x) \Rightarrow$ "f prime of x" $y' \Rightarrow$ "y prime"

$dy/dx \Rightarrow$ "d y-dx"

Examples:

$f(x) = x^2$, find f'(x). Then find f'(1).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

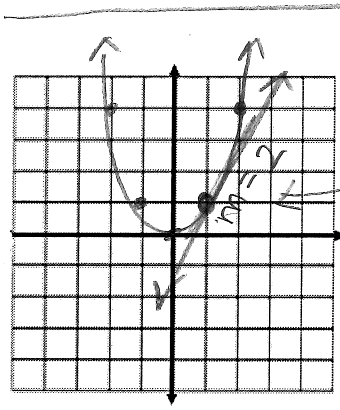
$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} (2x+h) = 2x$$

$$\boxed{f'(x) = 2x}$$

$$\boxed{f'(1) = 2 \cdot 1 = 2}$$

slope at $x=1$ is 2



check continuity first L: $f(0) = 0^2 = 0$

R: $f(0) = 5 \cdot 0 = 0$

same, so continuous

$$f(x) = \begin{cases} x^2, & x < 0 \\ 5x, & x \geq 0 \end{cases}$$

Need to find L & R derivatives

Find f'(0).

Left: $f'(x) = 2x$

$f'(0) = 2 \cdot 0 = 0$

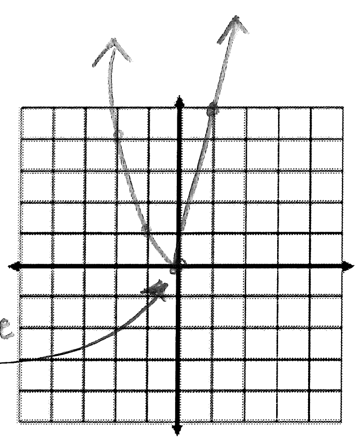
$$\boxed{f'(0) = DNE}$$

b/c these are different

Right: linear, so slope = 5 everywhere

$f'(x) = 5$

$f'(0) = 5$



Sharp corner here (2 different slopes)

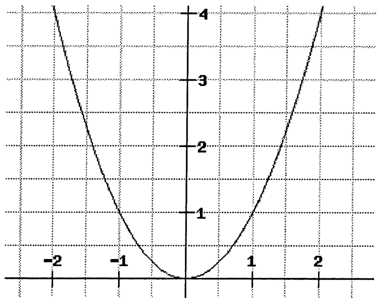
Derivative on Calculator: Math → 8 nDeriv(f(x), x, a)

function variable (type x) value of x

<u>Graphs of functions vs. Graphs of their derivatives:</u>

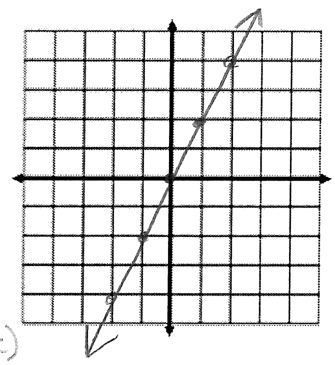
$f(x) = x^2$

ordered pairs (x, f(x))



x	f(x)	f'(x)
−2	4	−4
−1	1	−2
0	0	0
1	1	2
2	4	4

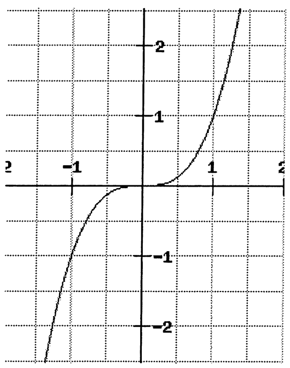
slopes of f(x) at particular x values



f'(x) = 2x

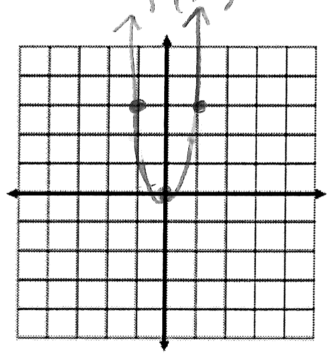
order pairs (x, f'(x))

$f(x) = x^3$



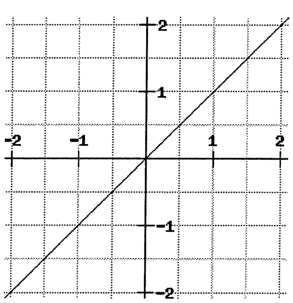
x	f(x)	f'(x)
−2	−8	12
−1	−1	3
0	0	0
1	1	3
2	8	12

f'(x)



On Calculator:
$Y_1 =$ nDeriv(x^3, x, x)
graph for all values of x

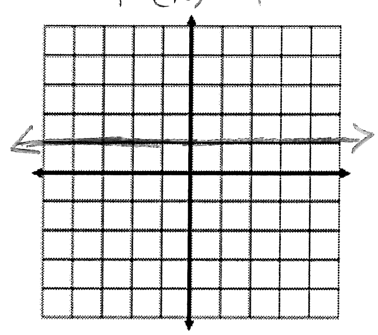
$f(x) = x$



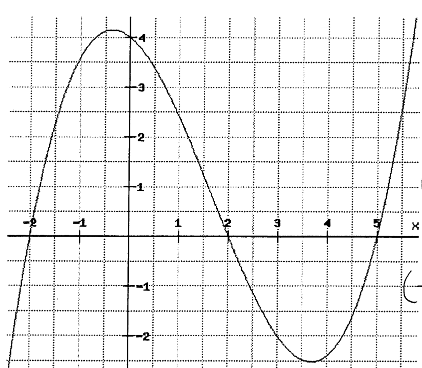
x	f(x)	f'(x)
−2	−2	1
−1	−1	1
0	0	1
1	1	1
2	2	1

slope always = 1

f'(x) = 1



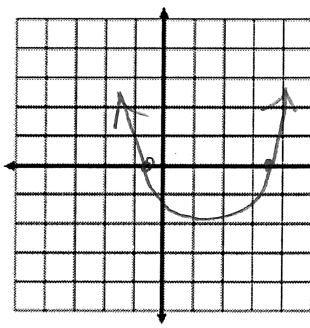
f(x)



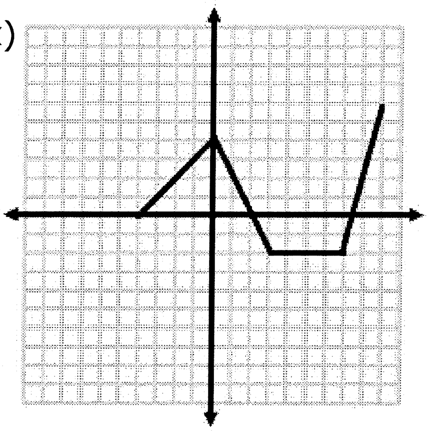
estimating

x	f'(x)
(−∞, −.5)	+
−.5	0
(−.5, 3.5)	−
3.5	0
(3.5, ∞)	+

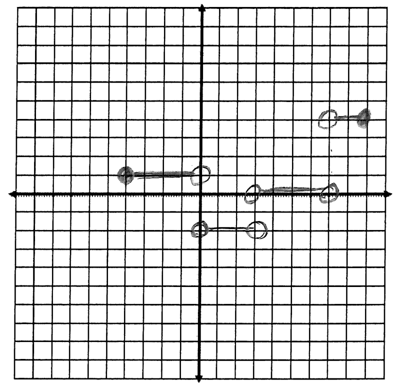
f'(x) ≈ something like this



f(x)

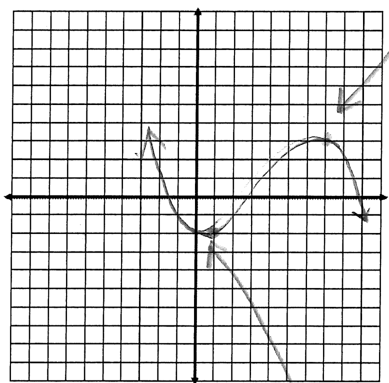


x	f′(x)
[-4,0)	1
(0,3)	-2
(3,7)	0
(7,9]	4



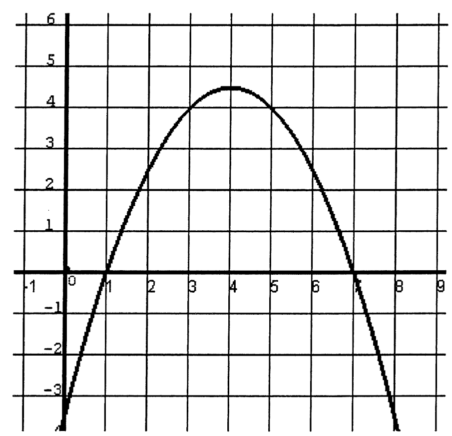
zero slope @x=7

f(x)



x	f′(x)
$(-\infty,1)$	neg
1	0
(1,7)	pos
7	0
$(7,\infty)$	neg

zero slope @x=1 (don't know y)



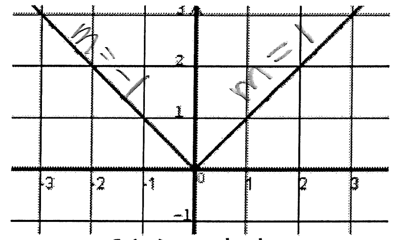
3.2 Differentiability

Four cases: $f'(x)$ does not exist at $x = a$.

1) Corner: left hand deriv. $\neq$ right hand deriv.

ex) $f(x) = |x|$

$\left.\begin{array}{l} \text{L deriv: } f'(0) = -1 \\ \text{R deriv: } f'(0) = 1 \end{array}\right]$ not same, not differentiable

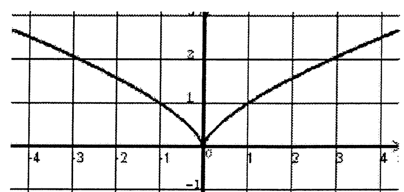


$f(x) = |x|$

2) Cusp: L deriv $\neq$ R deriv
($\pm\infty$) ($\mp\infty$)

ex) $f(x) = x^{2/3}$

$\left.\begin{array}{l} \text{L deriv: } f'(0) = -\infty \\ \text{R deriv: } f'(0) = +\infty \end{array}\right]$ not same (and not defined), not differentiable

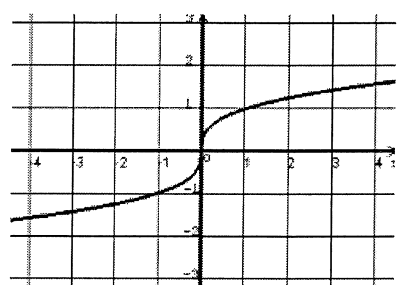


$f(x) = x^{2/3}$

3) Vertical tangent: L deriv = R deriv
($\pm\infty$) ($\pm\infty$)

ex) $f(x) = \sqrt[3]{x}$

$\left.\begin{array}{l} \text{L deriv: } f'(x) = +\infty \\ \text{R deriv: } f'(x) = +\infty \end{array}\right]$ "same", but neither exists so not differentiable

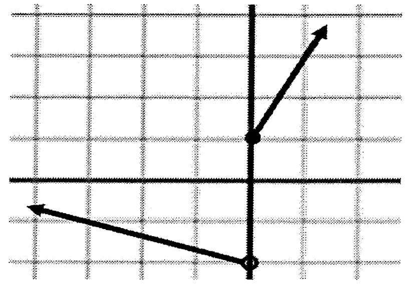


$f(x) = \sqrt[3]{x}$

4) Discontinuity: $\lim_{x \to a} f(x) \neq f(a)$

- jump
- point
- infinite
- oscillating

ex) $\left.\begin{array}{l} \lim_{x \to 0^-} f(x) = -2 \\ \lim_{x \to 0^+} f(x) = 1 \end{array}\right]$ $\lim_{x \to 0} f(x) = \text{DNE}$

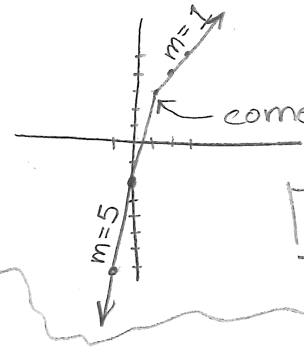


$$f(x) = \begin{cases} -\frac{1}{3}x - 2, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases}$$

$\left.\begin{array}{l} \lim_{x \to 0} f(x) = \text{DNE} \\ f(0) = 1 \end{array}\right]$ not same so discontinuous $\Rightarrow$ not differentiable

Example: Where do the following functions fail to be differentiable? Why? check graph on calc.

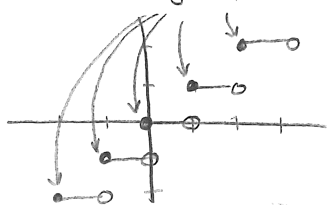
a) $y = 3x - 2|x - 1|$



corner here: left deriv. $\neq$ right deriv.
m=5 m=1

$\boxed{\text{not differentiable @ } x=1 \text{ b/c corner}}$

b) $y = [x]$



jumps @ x = k, k is an integer

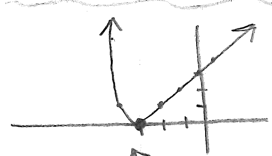
ex) x = 1

$\underbrace{\lim_{x\to 1} f(x)}_{DNE} \neq \underbrace{f(1)}_{1}$

$\boxed{\text{not differentiable @ } x=k \text{ b/c jumps}}$

c) $y = \begin{cases} x + 3, & x < -3 \\ (x + 3)^2, & x \geq -3 \end{cases}$ f(-3) = 0, f(-3) = 0 no jump

x = -3... suspicious
could be jump?
could be different slopes?

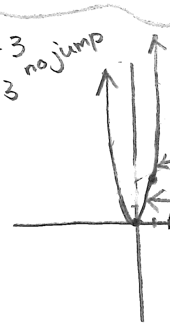


$\boxed{\text{not differentiable @ } x = -3 \text{ b/c corner}}$

corner here: left deriv $\neq$ right deriv.
m=0 m=1

d) $y = \begin{cases} 3x^2, & x \leq 1 \\ 2x^3 + 1, & x > 1 \end{cases}$ f(1) = 3, f(1) = 3 no jump

x = 1 ... suspicious
jump?
diff. slopes?



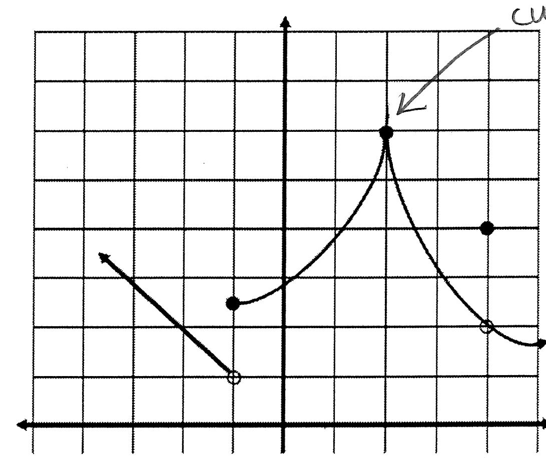
right deriv: at x=1 $\lim_{h\to 0} \frac{2(1+h)^3 + 1 - (2(1)^3 + 1)}{h} = 6$ trust me ☺

left deriv: at x=1 $\lim_{h\to 0} \frac{3(1+h)^2 - (3(1)^2)}{h} = 6$

$\boxed{\text{this function differentiable for all } x}$

L der = R der
no discont.

Example: At what points on [-2, 5] is f(x)...



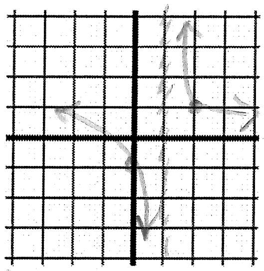
a) Differentiable?
[-2, -1), (-1, 2), (2, 4), (4, 5]

b) Continuous, but not differentiable?
x = 2 : cusp

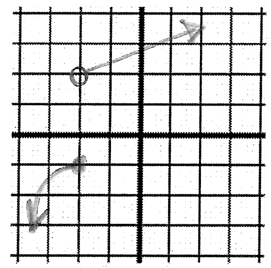
c) Neither continuous nor differentiable?
x = -1 : jump disc.
x = 4 : point disc.

Definition: A **differentiable function** is differentiable at all pts. in its domain

Differentiable at all pts. except x = 1 (VA), but x = 1 is <u>not</u> in domain, so f(x) is still called a "differentiable function"



$f(x) = \frac{1}{x-1}$



g(x)

differentiable at all pts except x = -2, domain includes x = -2 (f(-2) = -1) so g(x) is not a "differentiable function"

Theorem: Differentiability implies Continuity
If f has a derivative at x = a, then f is continuous at x = a

* but, continuous does <u>not</u> imply differentiability (cusp, corner, vert. tangent)

<u>Example:</u> $f'(-1) = 4$ and $f(-1) = 10$

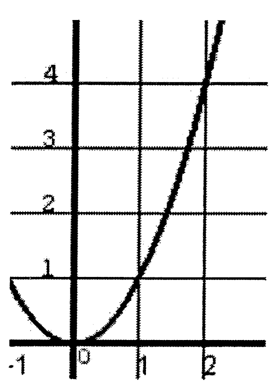
a) Is f continuous at x = -1? yes, b/c f'(-1) exists: differentiability ⇒ continuity

b) What is the equation of the tangent line at x = -1? pt = (-1, 10) slope = 4 $\boxed{y-10 = 4(x+1)}$

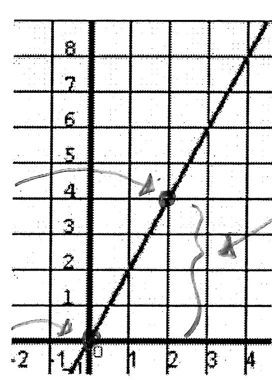
Theorem 2: Intermediate Value Theorem for Derivatives
If a and b are points on an interval where f is differentiable, then f ' takes all values between f '(a) and f '(b)

f(x) is differentiable on [0,2]

$f(x) = x^2$



f '(x) = 2x

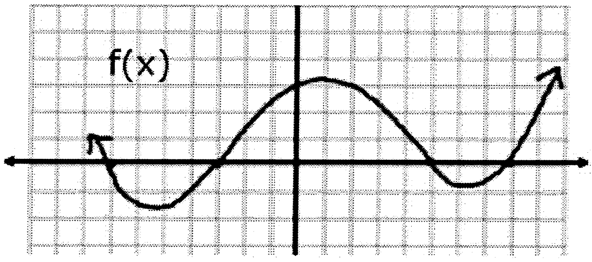


f '(2)

f '(0)

f ' takes all values between f '(0) and f '(2)

Examples:



On which intervals is f'(x) negative?

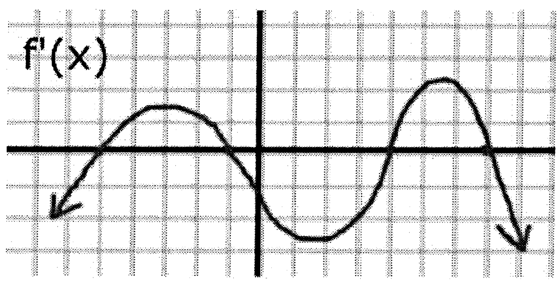
$(-\infty,-5),(1,6)$

b/c f(x) is decreasing

Rank from least to greatest:
f'(-5), f(0), f'(5)

slope @ x=-5 is zero f(0)=3 slope @ x=5 is negative

$f'(5), f'(-5), f(0)$



For what values of x does f(x) have a horizontal tangent?

x = -5, -1, 4, 7 b/c f' = 0

For what values of x on [-6, 8] does f(x) have a relative maximum?

min @ x = -1 and x = 7

b/c f' changes from + to −